A Reduced Model of ExB and PV Staircase Formation and Dynamics

P. H. Diamond,

M. Malkov,

A. Ashourvan; UCSD

D.W. Hughes; Leeds

U.S. DEPARTMENT OF ENERGY

UCSD

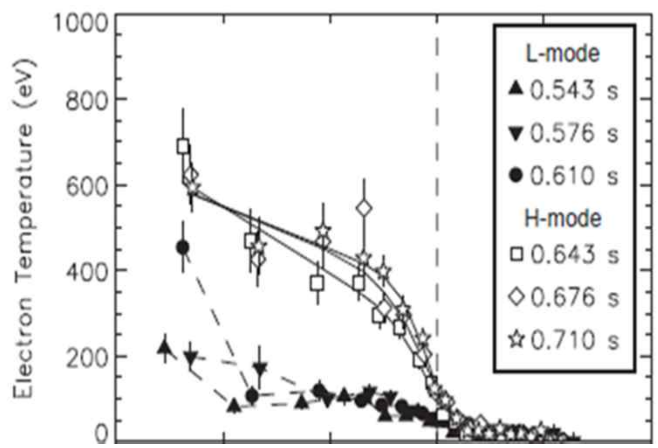
Outline

- Basic Ideas: transport bifurcation and ‘negative diffusion’ phenomena
  - Inhomogeneous mixing: times and lengths
  - Feedback loops
  - Simple example
- Inhomogeneous mixing in space:
  - Staircase models in QG and drift wave system
  - Models, background
  - QG staircase: model and results
  - Aside: on PV mixing
  - Model for Hasegawa-Wakatani system – 2 field, and intro
- Some Lessons and Conclusions

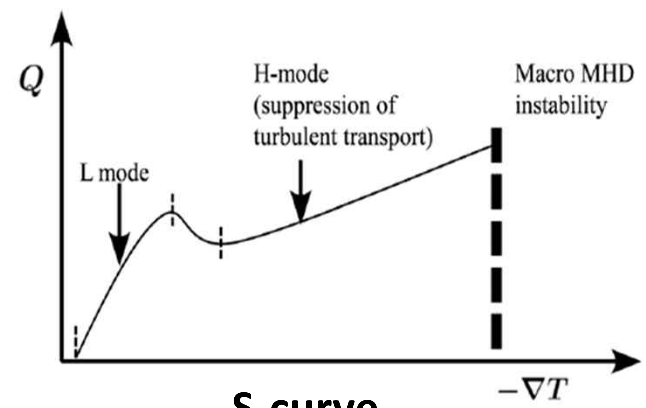
I) Basic Ideas:
Transport bifurcations and
‘negative diffusion’ phenomena

Transport Barrier Formation (Edge and Internal)

- Observation of ETB formation (L→H transition)
  - THE notable discovery in last 30 yrs of MFE research
  - Numerous extensions: ITB, I-mode, etc.
  - Mechanism: turbulence/transport suppression by ExB shear layers generated by turbulence
- Physics:
  - Spatio-temporal development of bifurcation front in evolving flux landscape
  - Cause of hysteresis, dynamics of back transition
- Fusion:
  - Pedestal width (along with MHD) → ITER ignition, performance
  - ITB control → AT mode
  - Hysteresis + back transition → ITER operation



J.W. Huges et al., PSFC/JA-05-35



S-curve

Why Transport Bifurcation? BDT '90, Hinton '91

- Sheared $V_{E\times B}$ flow quenches turbulence, transport ➔ intensity, phase correlations
- Gradient + electric field ➔ feedback loop

i.e. $\vec{E} = \frac{\nabla P_i}{nq} - \vec{V} \times \vec{B}$ ➔ $V_E' = V_E'(\nabla T)$

➔ minimal model $$Q = -\frac{\chi(\nabla T)\nabla T}{\left[1+\left(\frac{V_E'}{\omega_{eff}}\right)^2\right]^n} - \chi_{neo}\nabla T$$

turbulent transport
+ shear suppression

Residual collisional

n ≡ quenching exponent

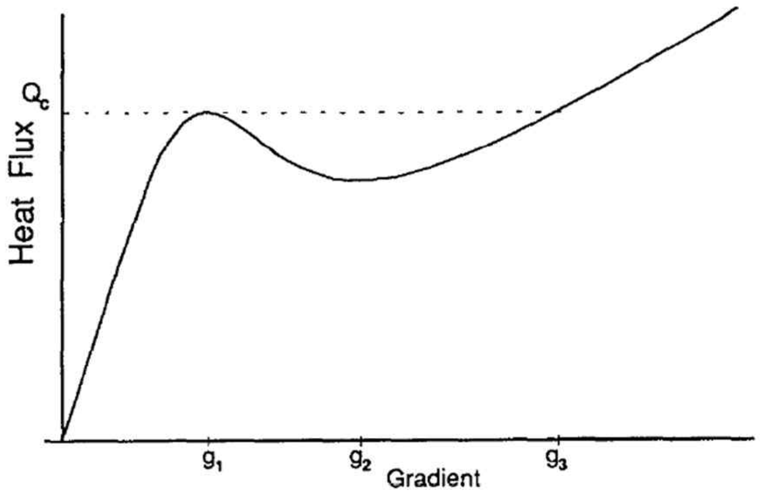
- Feedback:

$Q \uparrow \rightarrow \nabla T \uparrow \rightarrow V_E' \uparrow \rightarrow (\tilde{n}/n)^2, \chi_T \downarrow$

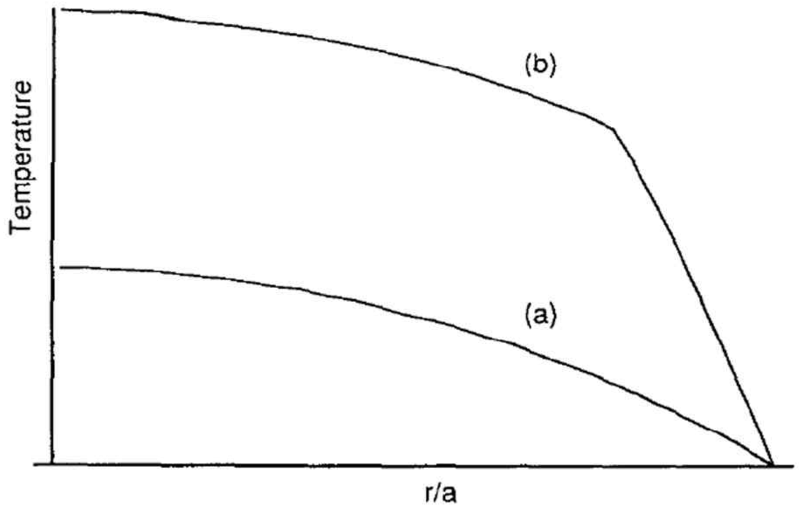
$\rightarrow \nabla T \uparrow \rightarrow \ldots$

- Result:

1st order transition (L→H):



Heat flux vs ∇T_i



T profiles
a) L-mode
b) H-mode

- S curve ➔ "negative diffusivity" i.e. $\delta Q \, / \, \delta \nabla T < 0$
- Transport bifurcations observed and intensively studied in MFE since 1982 yet:

➔ Little concern with staircases

➔ Key questions:

1) Might observed barriers form via step coalescence in staircases?
2) Is zonal flow pattern really a staircase (see GDP)?

Staircase in Fluids

- What is a staircase? – <u>sequence</u> of transport barriers
- Cf Phillips'72:

(other approaches possible)

SHORTER CONTRIBUTION

Turbulence in a strongly stratified fluid—is it unstable?

O. M. PHILLIPS*

(*Received* 30 *July* 1971; *in revised form* 6 *October* 1971; *accepted* 6 *October* 1971)

Abstract—It is shown that if the buoyancy flux is a local property of turbulence in a stratified fluid that decreases sufficiently rapidly as the local Richardson number increases, then an initially linear density profile in a turbulent flow far from boundaries may become unstable with respect to small variations in the vertical density gradient. An initially linear profile will then become ragged; this possible instability may be associated on occasions with the formation of density microstructure in the ocean.

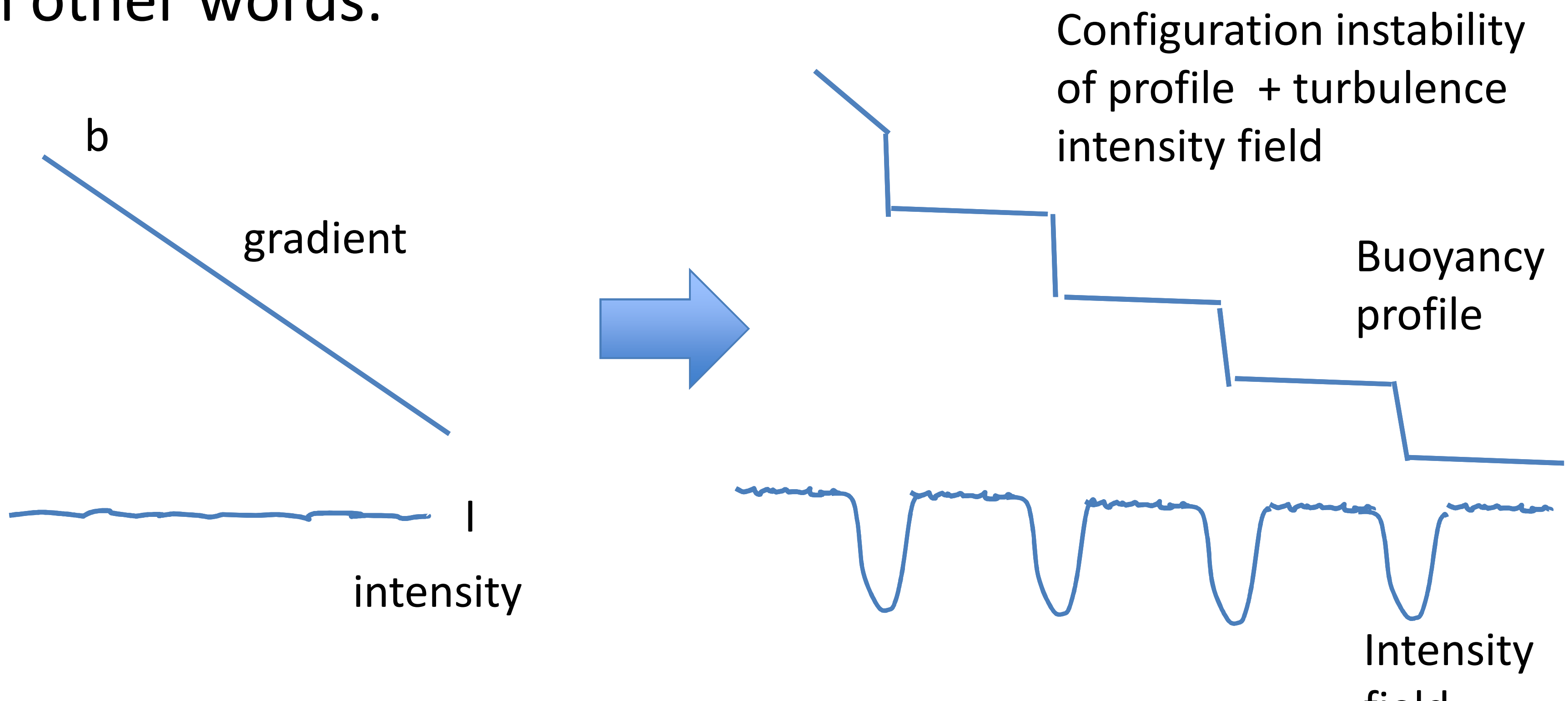
- Instability of mean + turbulence field requiring:

  $\delta\Gamma_b/\delta Ri < 0$; flux dropping with increased gradient

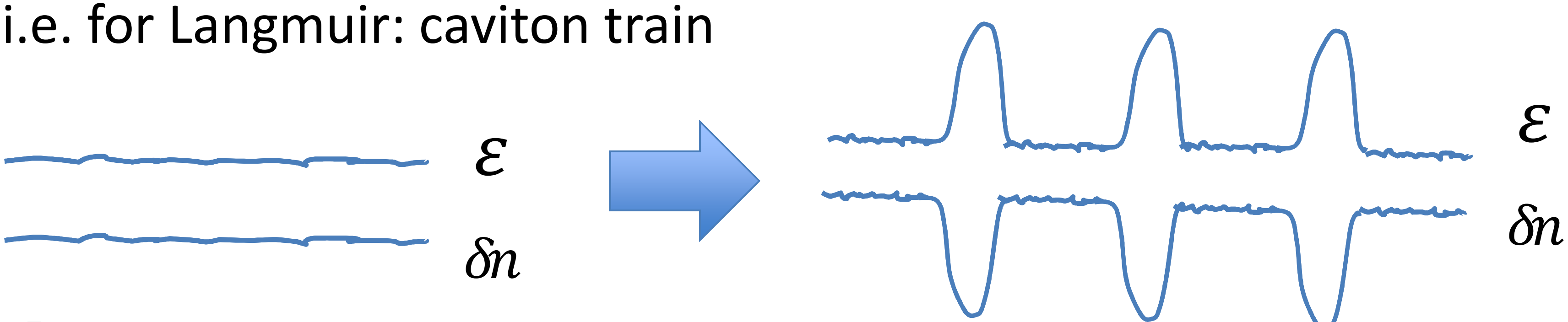
  $\Gamma_b = -D_b \nabla b, Ri = g\nabla b/(v')^2$

- Obvious similarity to transport bifurcation

In other words:



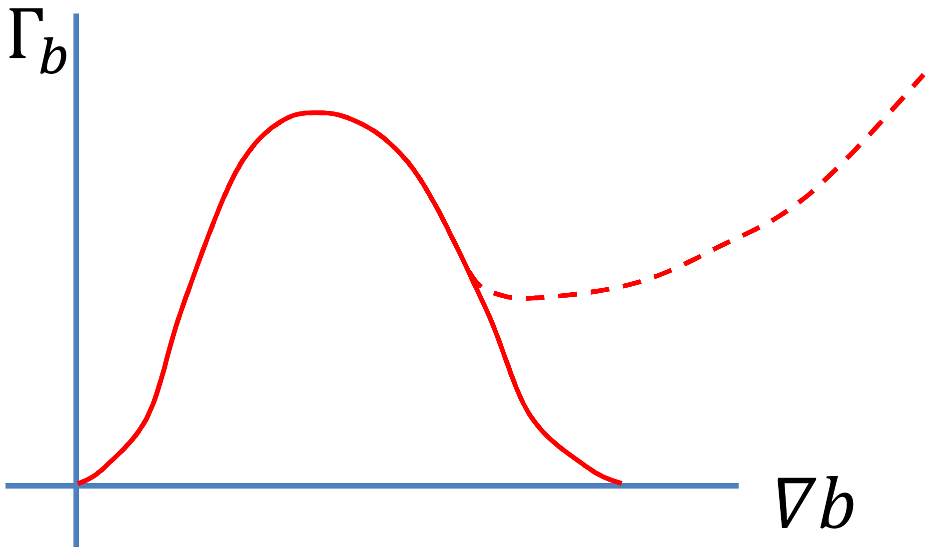
Some resemblance to Langmuir turbulence
i.e. for Langmuir: caviton train



➔ end state of
<u>modulational instability !?</u>

$$\delta n / n \approx -\varepsilon$$

- The physics: Negative Diffusion (BLY, ‘98)



“H-mode” like branch
(i.e. residual collisional diffusion)
is not input

- Usually no residual diffusion
- ‘branch’ upswing → nonlinear processes (turbulence spreading)
- If significant molecular diffusion → second branch

- Instability driven by local transport bifurcation

➔ • $\delta\Gamma_b/\delta\nabla b < 0$

➔ ‘negative diffusion’

Negative slope
Unstable branch

➔ • Feedback loop $\Gamma_b \downarrow \rightarrow \nabla b \uparrow \rightarrow I \downarrow \rightarrow \Gamma_b \downarrow$

Critical element:
$l \rightarrow$ mixing length

The Critical Element: Mixing Length

- Sets range of inhomogeneous mixing

- $\frac{1}{l^2} = \frac{1}{l_0^2} + \frac{1}{l_{oz}^2}$

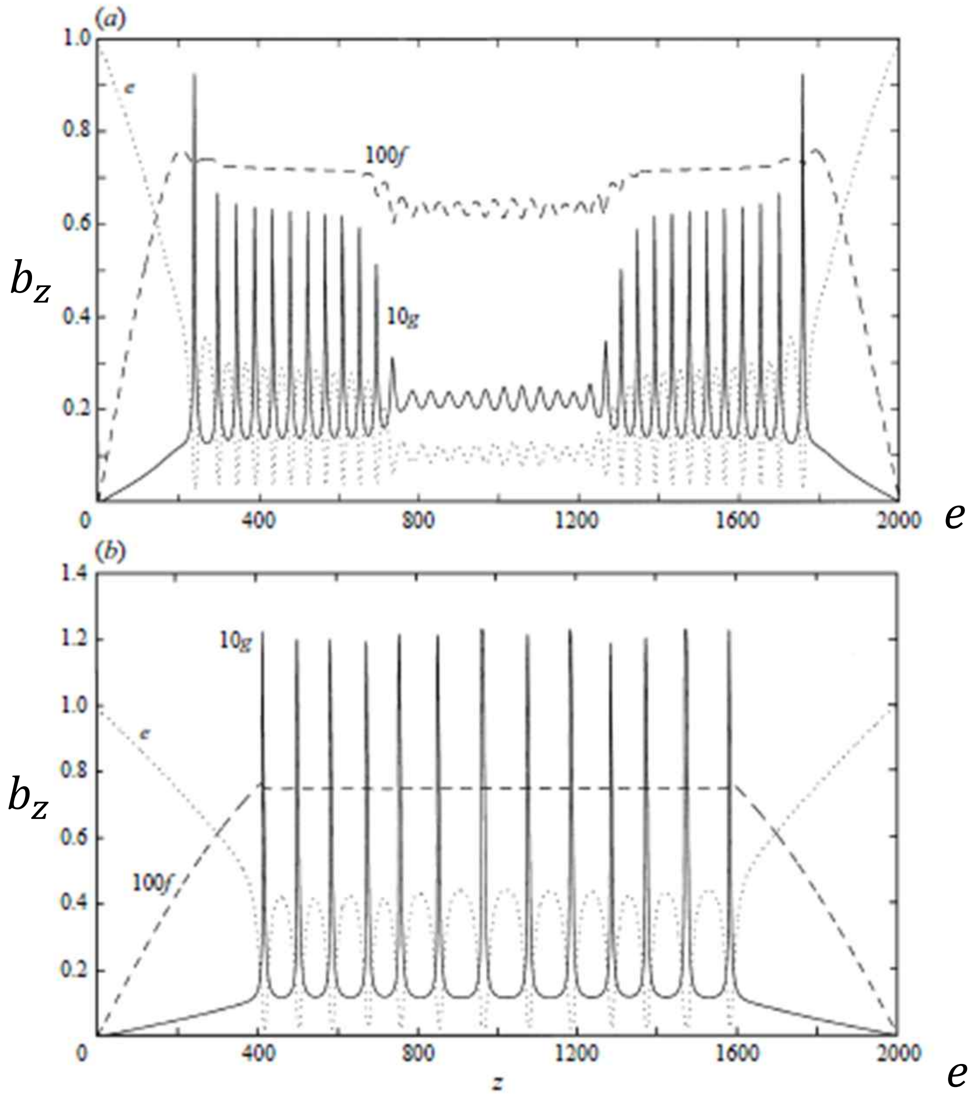
- l_{oz} ~ Ozmidov scale, smallest 'stratified scale'

  ←→ balance of buoyancy and production

- $\frac{1}{l_{oz}} \approx \left(\frac{b_z}{e}\right)^{\frac{1}{2}}$ ➔ b_z dependence is crucial for inhomogeneous mixing

- Feedback loop: $b_z \uparrow$ → $e \downarrow$ → $l \downarrow$

- <u>A Few Results</u> → $\nabla\rho$ staircases



Plot of b_z (solid) and e (dotted) at early time. Buoyancy flux is dashed → near constant in core

Later time → more akin expected "staircase pattern". Some <u>condensation</u> into larger scale structures has occurred.

II) Inhomogeneous Mixing in Space: Staircase Models in QG and Drift Wave Systems

Drift wave model – Fundamental prototype

- Hasegawa-Wakatani : simplest model incorporating instability

$$V = \frac{c}{B}\hat{z} \times \nabla\phi + V_{pol}$$

$$J_\perp = n|e|V^i_{pol} \qquad \eta J_\parallel = -\nabla_\parallel \phi + \nabla_\parallel p_e$$

$$\nabla_\perp \cdot J_\perp + \nabla_\parallel J_\parallel = 0$$ → vorticity: $\rho_s^2 \frac{d}{dt}\nabla^2\phi = -D_\parallel \nabla_\parallel^2(\phi - n) + \nu\nabla^2\nabla^2\phi$

$$\frac{dn_e}{dt} + \frac{\nabla_\parallel J_\parallel}{-n_0|e|} = 0$$ → density: $\frac{d}{dt}n = -D_\parallel \nabla_\parallel^2(\phi - n) + D_0\nabla^2 n$

→ **PV conservation in inviscid theory** $\frac{d}{dt}\left(n - \nabla^2\phi\right) = 0$

→ PV flux = particle flux + vorticity flux

→ zonal flow being a counterpart of particle flux

QL: $\frac{\partial}{\partial t}\langle n\rangle = -\frac{\partial}{\partial r}\langle \tilde{v}_r \tilde{n}\rangle$

→? $\frac{\partial}{\partial t}\langle \nabla^2\phi\rangle = -\frac{\partial}{\partial r}\langle \tilde{v}_r \nabla^2\tilde{\phi}\rangle = -\frac{\partial^2}{\partial r^2}\langle \tilde{v}_r \tilde{v}_\theta\rangle$

- Hasegawa-Mima ($D_\parallel k_\parallel^2/\omega >> 1 \rightarrow n \sim \phi$)

$$\frac{d}{dt}\left(\phi - \rho_s^2\nabla^2\phi\right) + \upsilon_* \partial_y \phi = 0$$

- **Key: PV conservation dq/dt=0**

GFD: Quasi-geostrophic system	Plasma: Hasegawa-Wakatani system
$q = \nabla^2 \psi + \beta y$ $\nabla^2\psi$ → relative vorticity; βy → planetary vorticity	$q = n - \nabla^2 \phi$ n → density (guiding center); $\nabla^2\phi$ → ion vorticity (polarization)
Physics: $\Delta y \rightarrow \Delta\left(\nabla^2 \psi\right)$	Physics: $\Delta r \rightarrow \Delta n \rightarrow \Delta\left(\nabla^2 \phi\right)$ ZF!

- Charney-Haswgawa-Mima equation

$$n = n_0 + \tilde{n}$$

$$\tilde{n} \sim \frac{e\tilde{\phi}}{T}$$

H-W → H-M: $$\frac{1}{\omega_{ci}} \frac{\partial}{\partial t}\left(\nabla^2 \phi - \rho_s^{-2} \phi\right) - \frac{1}{L_n} \frac{\partial}{\partial y} \phi + \frac{\rho_s}{L_n} J(\phi, \nabla^2 \phi) = 0$$

Q-G: $$\frac{\partial}{\partial t}\left(\nabla^2 \psi - L_d^{-2} \psi\right) + \beta \frac{\partial}{\partial x} \psi + J(\psi, \nabla^2 \psi) = 0$$

Staircase in QG Turbulence: A Model

- PV staircases observed in nature, and in the unnatural
- Formulate 'minimal' dynamical model ?! (n.b. Dritschel-McIntyre 2008 does not address dynamics)

Observe:

- 1D adequate: for ZF need 'inhomogeneous PV mixing' + 1 direction of symmetry. Expect ZF staircase
- Best formulate intensity dynamics in terms potential enstrophy $\epsilon = \langle \tilde{q}^2 \rangle$
- Length? : $\Gamma_q \partial \langle q \rangle / \partial y \sim \tilde{q}^3$ (production-dissipation balance)
- → $l \sim \langle \tilde{q}^2 \rangle^{\frac{1}{2}} / \partial \langle q \rangle / \partial y \sim l_{Rhines}$ (i.e. $\omega_{Rossby} \sim k \tilde{v}$)

Model: $\Gamma_q = \langle \tilde{v}_y \tilde{q} \rangle = -D\partial\langle q \rangle/\partial y$ is fundamental quantity

➔ Mean: $\partial_t \langle q \rangle = \partial_y D \partial_y \langle q \rangle$

➔ Potential Enstrophy density: $\partial_t \epsilon - \partial_y D \partial_y \epsilon = D\left(\partial_y \langle q \rangle\right)^2 - \epsilon^{\frac{3}{2}} + F$

Spreading (term $\partial_y D \partial_y \epsilon$), Production (term $D\left(\partial_y \langle q \rangle\right)^2$), Dissipation (term $\epsilon^{\frac{3}{2}}$), Forcing (term F)

Where:

$$\frac{1}{l^2} = \frac{1}{l_f^2} + \frac{1}{l_{Rh}^2}$$

$$l_{Rh}^2 = \epsilon / \left(\partial_y \langle q \rangle\right)^2$$

$D \sim l^2 \sqrt{\epsilon}$ (dimensional)

$$D_{spr} \approx D_{PV}$$

$\partial_t \left(\frac{\langle q \rangle^2}{2} + \epsilon \right) = 0$, to forcing, dissipation

Alternative Perspective:

- Note: $l^2 = \frac{1}{1+1/l_{Rh}^2} \rightarrow \frac{1}{1+\langle q \rangle'^2/\epsilon}$ $\quad (l_f \sim 1)$
- Reminiscent of weak turbulence perspective:

$$D = D_{pv} = \sum_{\vec{k}} \frac{\langle \tilde{V}^2 \rangle \Delta\omega_{\vec{k}}}{\omega_{\vec{k}}^2 + \Delta\omega_{\vec{k}}^2}$$

$$\omega_{\vec{k}} = -k_x \langle q \rangle' / k^2$$

$$\Delta\omega_{\vec{k}} \approx k\tilde{V}_{\vec{k}}$$

Ala' Dupree'67:

$$D_{pv} \approx \frac{1}{k^2} \left(\sum_{\vec{k}} k^2 \langle \tilde{V}^2 \rangle_{\vec{k}} - \frac{k_x^2 \left(\langle q \rangle' \right)^2}{(k^2)^2} \right)^{1/2}$$

Steeper $\langle q \rangle'$ quenches diffusion → barrier via <u>PV gradient</u> feedback

$$D_{pv} \approx \frac{l_0^2\, \epsilon^{\frac{1}{2}}}{1 + \frac{l_0^2}{\epsilon}(\langle q \rangle')^2} \quad \leftarrow$$

- ω vs $\Delta\omega$ dependence gives D_{pv} roll-over with steepening
- Rhines scale appears naturally, in feedback strength
- Recovers effectively same model

Physics:

① “Rossby wave elasticity’ (MM) → steeper $\langle q \rangle'$ → stronger memory (i.e. more ‘waves’ vs turbulence)

② Distinct from shear suppression → interesting to dis-entangle

Aside

- What of wave momentum? Austauch ansatz

  Debatable (McIntyre) - but l_{mix} (?)...

- PV mixing ←→ $D \partial_y \langle q \rangle$

  So → $\langle \tilde{V} \tilde{q} \rangle$ → $\partial_y \langle \tilde{V}_y \tilde{V}_x \rangle$ → R.S.

- But:

  R.S. ←→ $\langle k_x k_y \rangle$ ←→ $V_{gy} E$

➔ Feedback:

$$\langle q \rangle' \uparrow \rightarrow l \downarrow \rightarrow \epsilon \downarrow \rightarrow D \downarrow$$

(Production)

- Equivalent!
- Formulate in terms mean, Pseudomomentum?

* - Red herring for barriers
  → l_{mix} quenched

Numerical Results:
Analysis of QG Model

- Re-scaled system

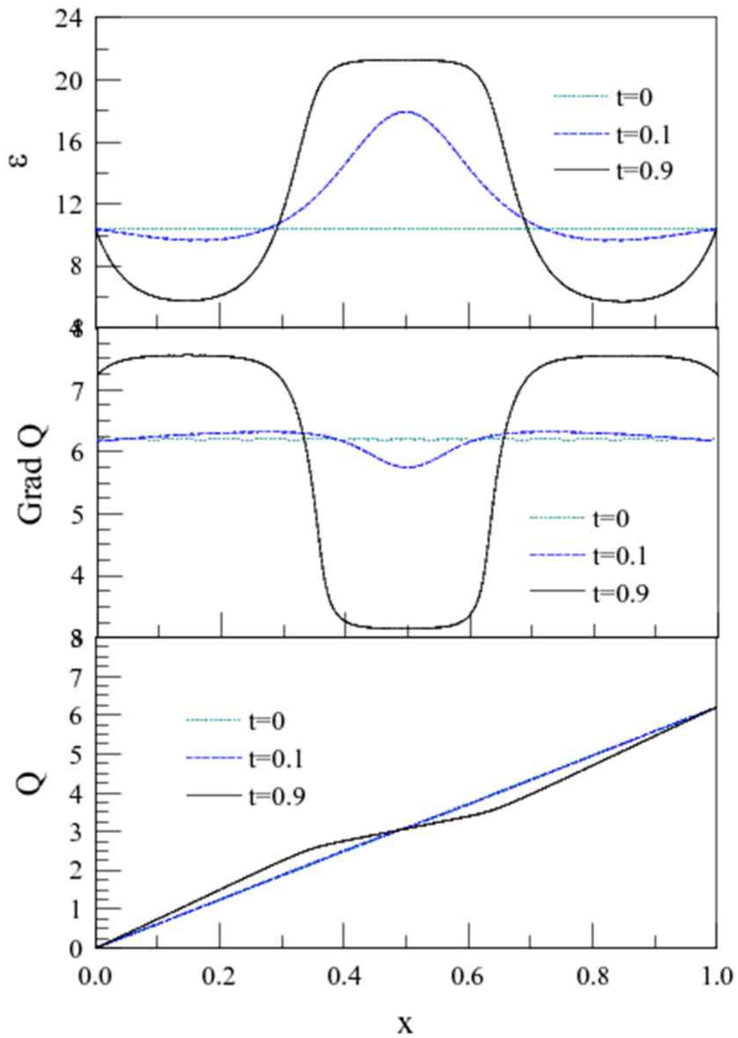
$$Q_t = \partial_y \frac{\varepsilon^{1/2}}{\left(1+Q_y^2\varepsilon\right)^{\kappa}}\, Q_y + D_{neo}\, Q_{yy} \quad \text{for mean}$$

$$\varepsilon_t = \partial_y \frac{\varepsilon^{1/2}}{\left(1+Q_y^2/\varepsilon\right)^{\kappa}}\, Q_y + L^2 \left\{ \frac{Q_y^2}{\left(1+Q_y^2/\varepsilon\right)^{\kappa}} - \frac{\varepsilon}{\varepsilon_0} + 1 \right\} \varepsilon^{1/2} \quad \text{for P.E.}$$

drive

- Note:
  - Quenching exponent $\kappa = 2$ for saturated modulational instability
  - Potential enstrophy conserved to forcing, dissipation, boundary
  - System size L ➔ strength of drive

- Weak Drive ➔ 1 step staircase

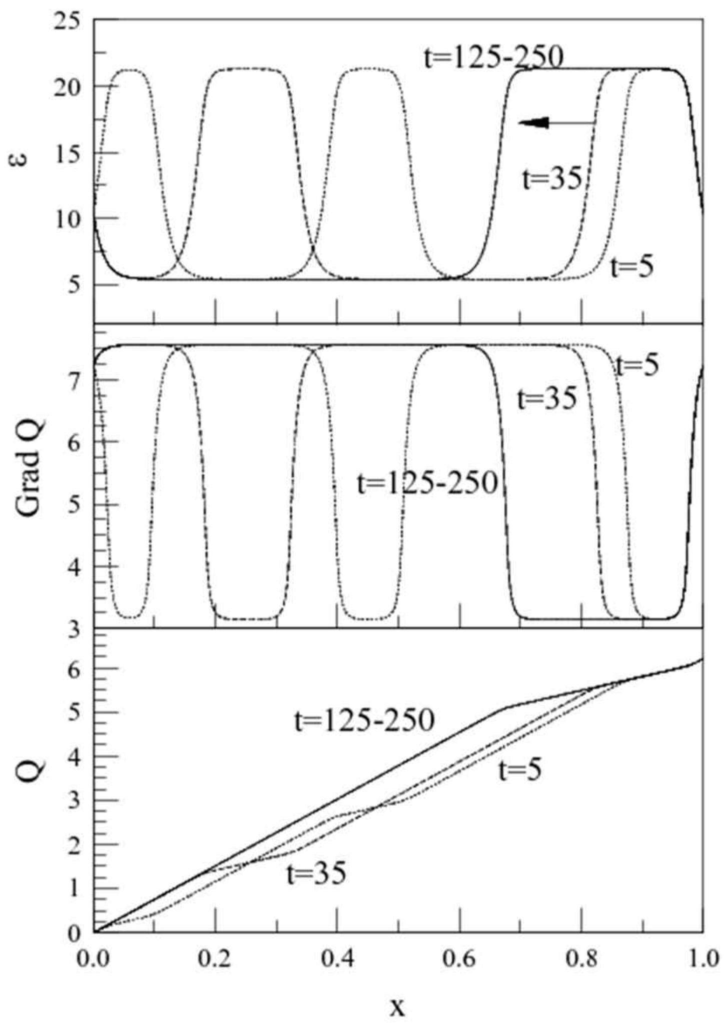


- 1 step staircase forms
- Small scales not evident
- Dirichlet B.C.'s

  Initial ΔQ

  } All

- Increased Drive ➔ Multi-step structure



– Multiple steps

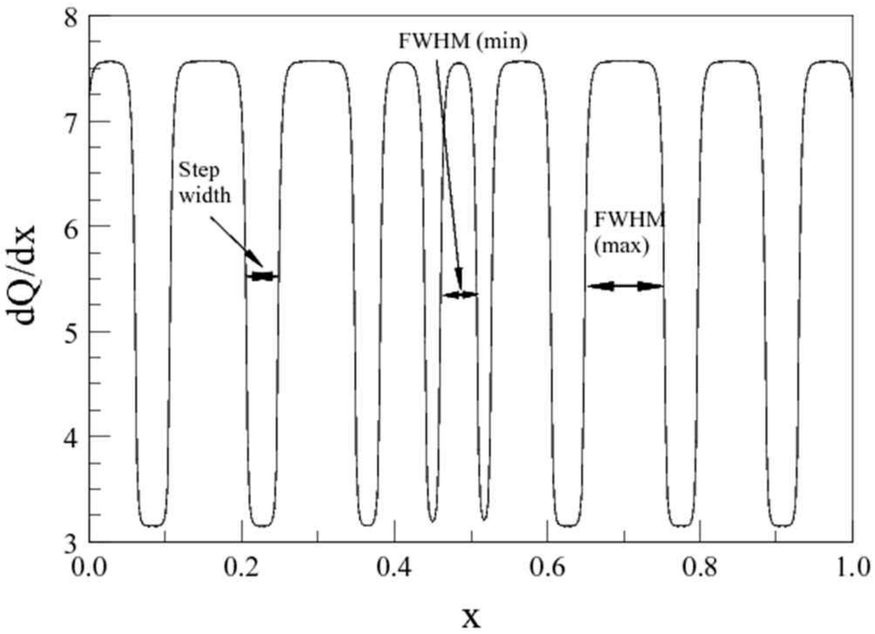
– Steps move

– Some hint of step condensation at foot of Q profile

* – End state: barrier on LHS, step on RHS

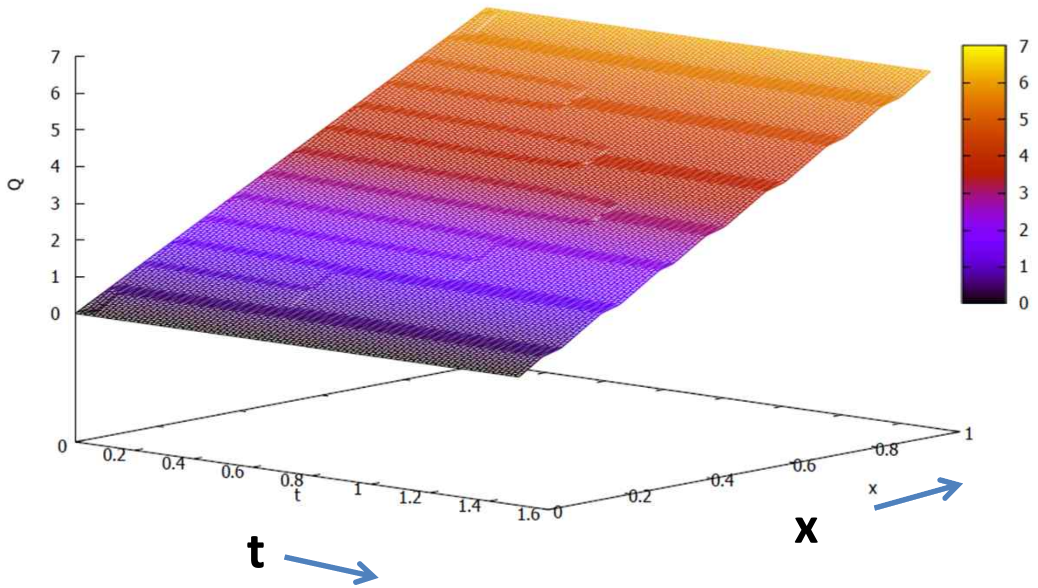
➔ Suggestive of barrier formation by staircase condensation

- ∇Q plot reveals structure and scales involved



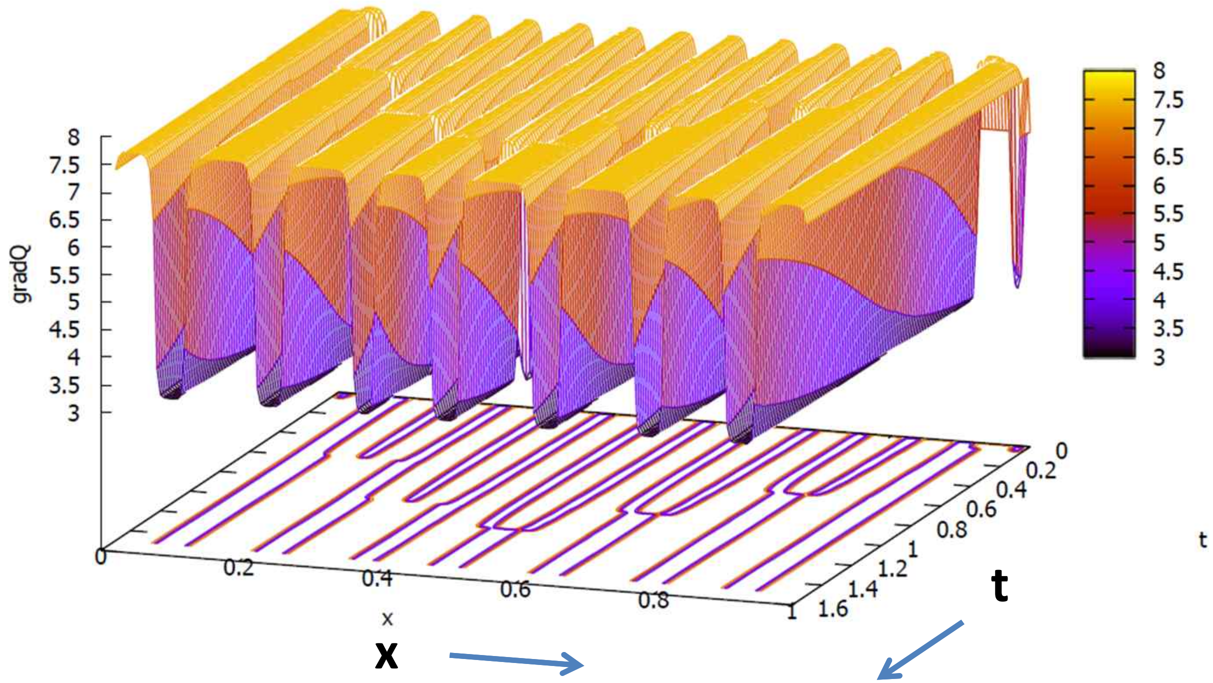
  - FW HM max, min capture width of steep gradient region
  - Step width - minimum

- Mergers occur



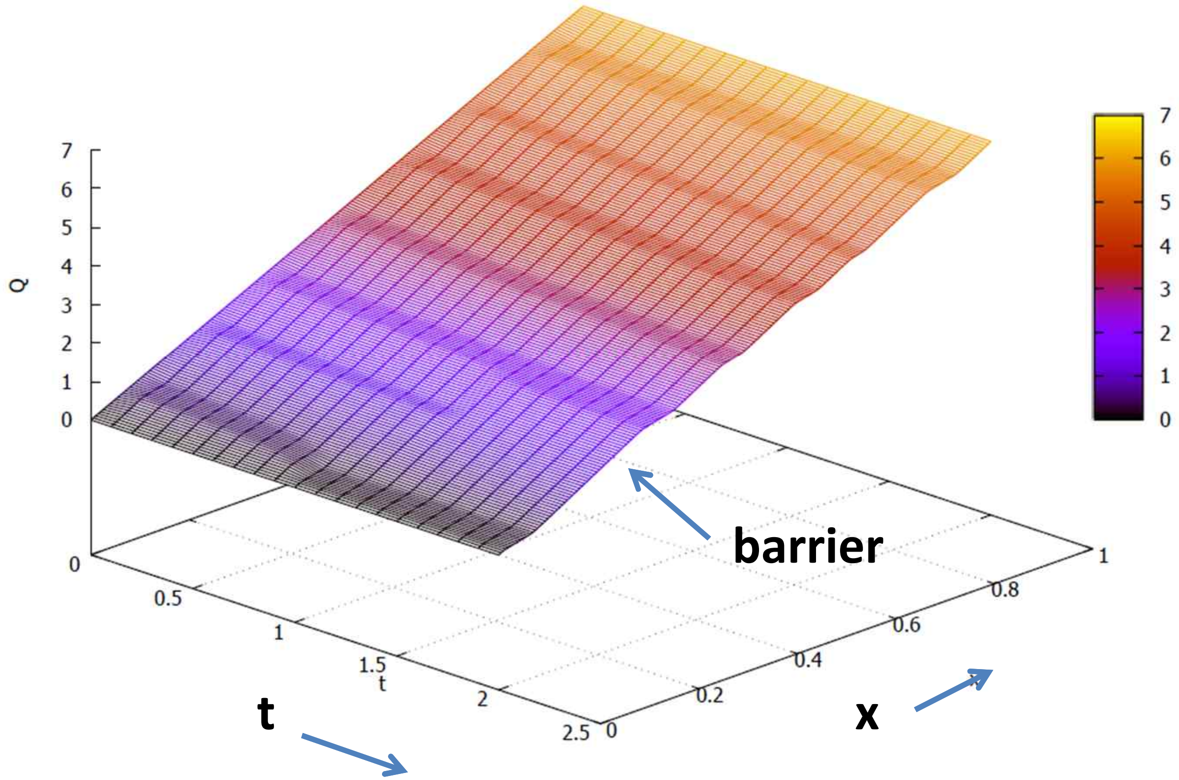
  - Same drive as before
  - Staircase smooths

- ∇Q plot of Mergers

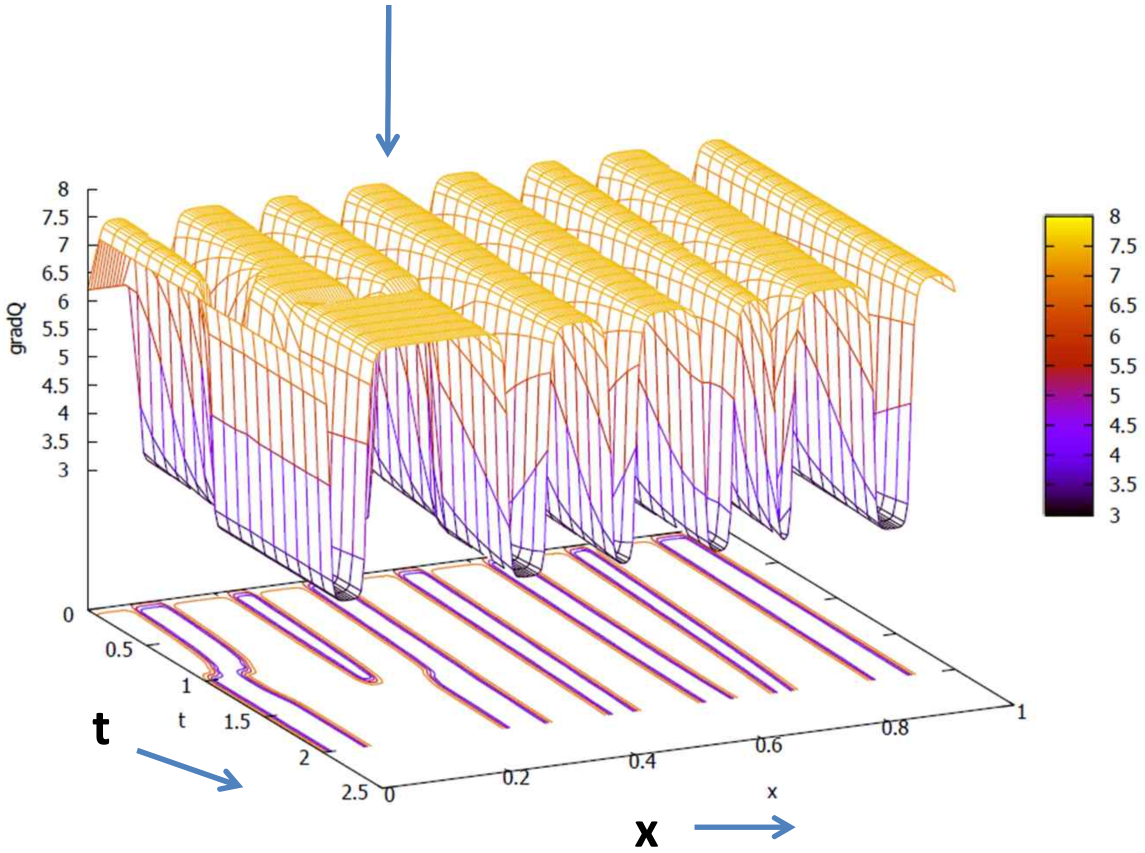


  - Can see region of peak ∇Q expanding
  - Coalescence of steps occurs
  - Some evidence for "bubble competition" behavior

- Mergers for yet stronger drive

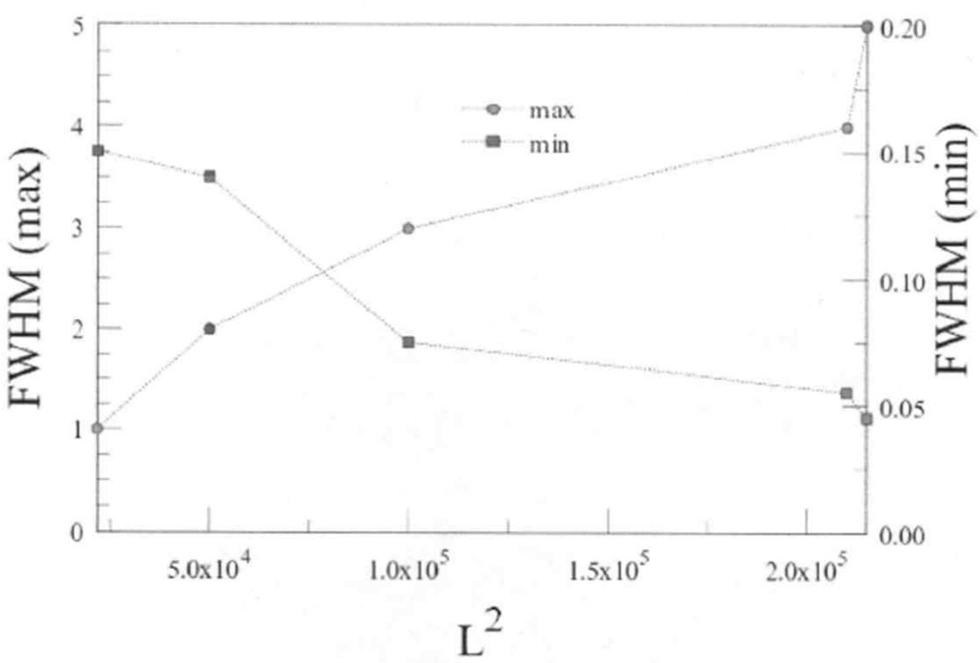


  - Mergers form broad barrier region at foot of Q profile
  - Extended mean barrier emerging from step condensation



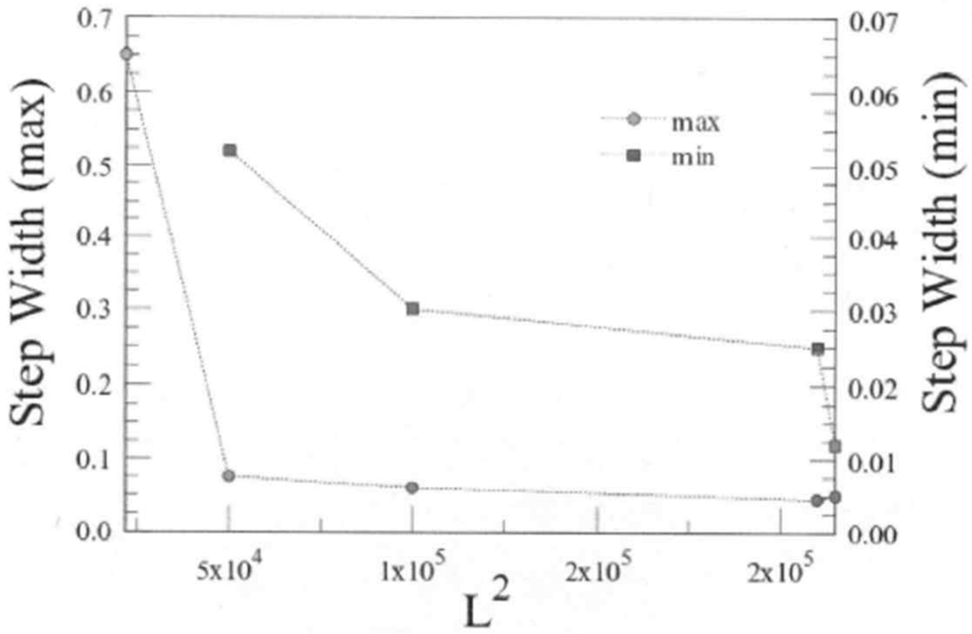
- ∇Q evolution in condensation process → same scale
- Note broadening of high ∇Q region near boundary

- Staircase Barrier Structure vs Drive



- L^2 ↑ ➔ increasing drive
- FW HM max increases with L^2, so
- Width of barrier expands as L^2 increases

- Staircase Step Structure vs Drive



  - Step width decreases and ~ saturated as L^2 increases
  - Min, max converge

- More interesting model…
  - From Hasegawa-Wakatani:

$$\frac{d}{dt}\nabla^2\phi = D_{\parallel}\nabla_{\parallel}^2(n-\phi) + \nu_0\nabla^2\nabla^2\phi$$

$$\frac{d}{dt}\mathrm{n} = \mathrm{D}_{\parallel}\nabla_{\parallel}^2(n-\phi) + D_0\nabla^2 n$$

$$\frac{\partial}{\partial t} + \vec{V}\cdot\nabla = \frac{d}{dt}; \quad \frac{k_{\parallel}^2 D_{\parallel}}{\omega} > 1; \quad \nu_0 > D_0$$

  - Evident that mean-field dynamics controlled by:
  - $\Gamma_n = \langle\tilde{v}_r\tilde{n}\rangle$ → particle flux
  - $\Gamma_u = \langle\tilde{v}_r\nabla^2\tilde{\phi}\rangle$ → vorticity flux

} Relation of ∇n corrugations and shear layers

- K-ϵ Model

$$u = \nabla^2 \phi$$

$$\left.\begin{aligned} &\partial_t n + \partial_x \Gamma_n = D_0 \partial_x^2 n \\ &\partial_t u + \partial_x \Gamma_u = \nu_0 \partial_x^2 u \end{aligned}\right\} \text{ mean}$$

$$\varepsilon = \text{Pot Enstr} = \langle \left(\tilde{n} - \nabla^2 \tilde{\phi}\right)^2 \rangle$$

$$\partial_t \varepsilon + \partial_x \Gamma_\varepsilon = -(\Gamma_n - \Gamma_u)(\partial_x n - \partial_x u) - \varepsilon^{\frac{3}{2}} + f$$

- Total P.E. conserved, manifestly
- $\Gamma_\varepsilon = \langle v_r \varepsilon \rangle$ → spreading flux
- Forcing as linear stage irrelevant

- Fluxes Γ_n, Γ_u
  - Could proceed as before ➔ PV mixing with feedback for steepened ∇q
  - i.e. $\Gamma_n = -D_T \partial_x n$
  - $\Gamma_u = -D_T \partial_x u$
  - $D_T \sim l_{mix}^2 \, (\varepsilon)^{1/2}$, with $1/l_{mix} = 1/l_0^2 + 1/l_{Rh}^2$
  - $l_{mix}^2 = l_0^2 \varepsilon \, / [\varepsilon + l_0^2 \, (\partial_r (n - u))^2]$

➔

- Fluxes Γ_n, Γ_u
  - Could proceed as before ➔ PV mixing with feedback for steepened ∇q
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  - $l_{mix}^2 = l_0^2 \varepsilon / [\varepsilon + l_0^2 (\partial_r (n-u))^2]$

➔Feedback by ∇q steepening and reduced D_T etc

➔Barrier structure?!

- More interesting: As CDW turbulence is wave turbulence, use mean field/QL theory as guide to model construction
- For QL theory, see Ashourvan, P.D., Gurcan (2015)
- Simplified:

(a) $\Gamma_n \approx -D_n \partial_x n$

$$D_n = -\langle \tilde{v}_r^2 \rangle \tau_c, \quad \tau_c^{-1} = \langle k_\parallel^2 D_\parallel \rangle$$

- Key: electron response laminar
- Neglected weak particle pinch

(b) $\Gamma_u = -\chi_y \nabla u + \Pi^{resid}$

$\chi_y \approx \langle \tilde{v}_r^2 \rangle (\gamma_k / (\omega - k_\theta v_\theta)^2) \rightarrow \langle \tilde{v}_r^2 \rangle / |u|$

$\Pi^{resid} \approx \Gamma_u / n_0 - \chi_y v_d \qquad v_d = -\partial_r n$

And $\langle \tilde{v}_r^2 \rangle \sim l_{mix}^2 \, \varepsilon$

N.B.: In QLT, $D_n \neq \chi_y$

- Interesting to note varied roles of:
  - Transport coefficients D_n, χ
  - Non-diffusive stress
  - Length scale, suppression exponent
  - Intensity dependence

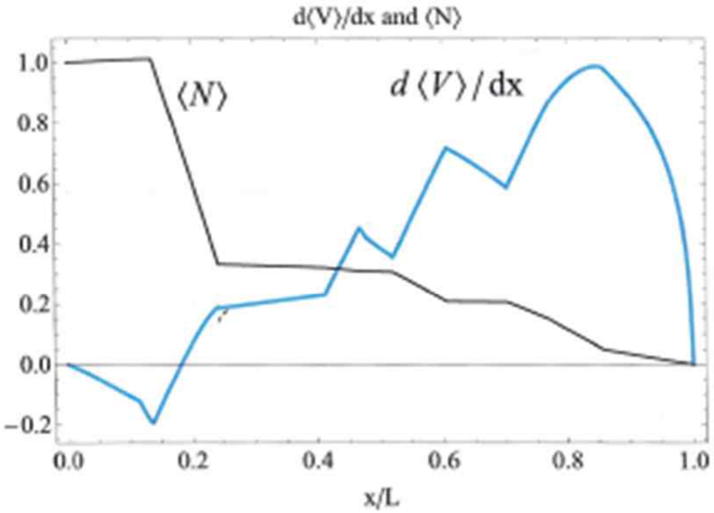
- Studies so far:
  - $D_n = D_u$ with ∇q feedback as in QG via l_{mix}

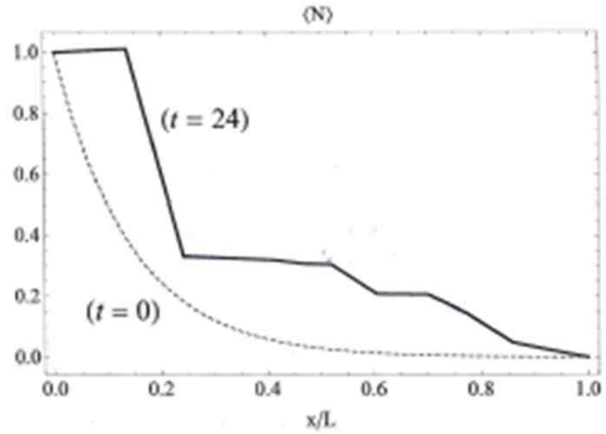
    $\kappa = 1, 2$
  - QL model with $l_{mix}(\nabla q)$

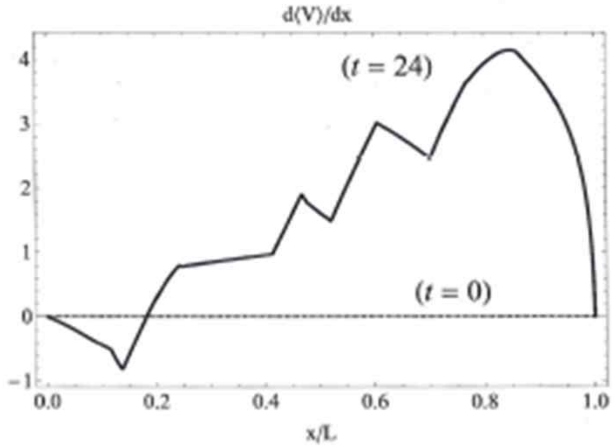
    $\kappa = 1,2$
- → these constitute perhaps the simplest cases conceivable…

- $l_{mix}^{-2} = l_0^{-2} + l_{Rh}^{-2}$, $D_n = D_u$ → mixing

Kappa=1



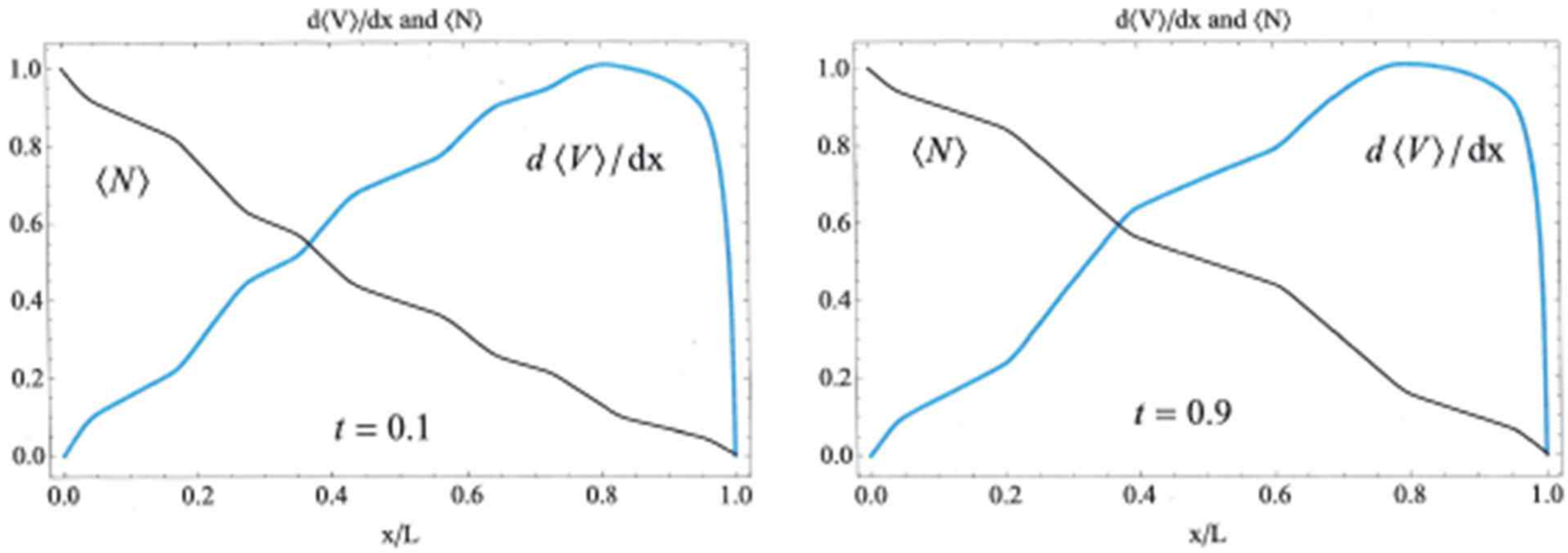




- Barrier and irregular staircase form
- Shear layer self-organizes near boundary

- $l_{mix}^{-2} = l_0^{-2} + l_{Rh}^{-2}$, $D_n = D_u$ → mixing

$\kappa = 2$

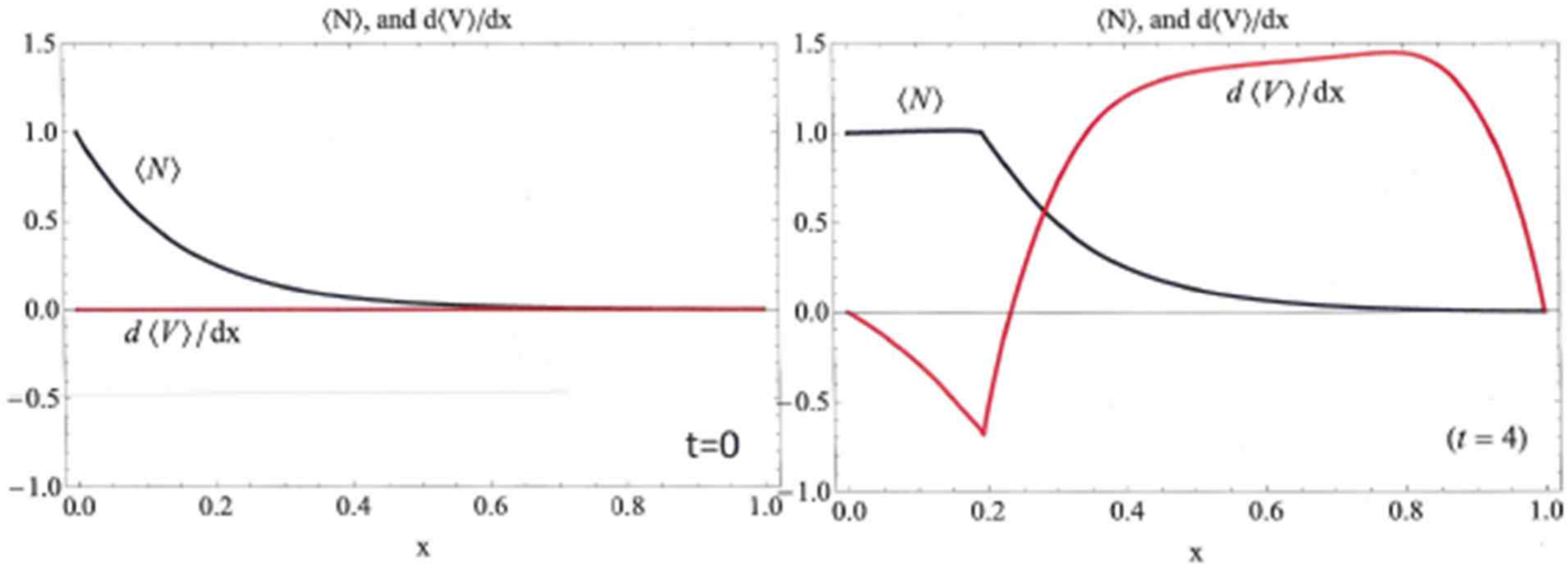


- Density and vorticity staircase form
- Regular in structure
- Condensation to large steps, barrier forms

- Quasilinear with l_{mix} feedback

$D_n \neq \chi_y$, $\Pi_{resid} \neq 0$

$\kappa = 1$

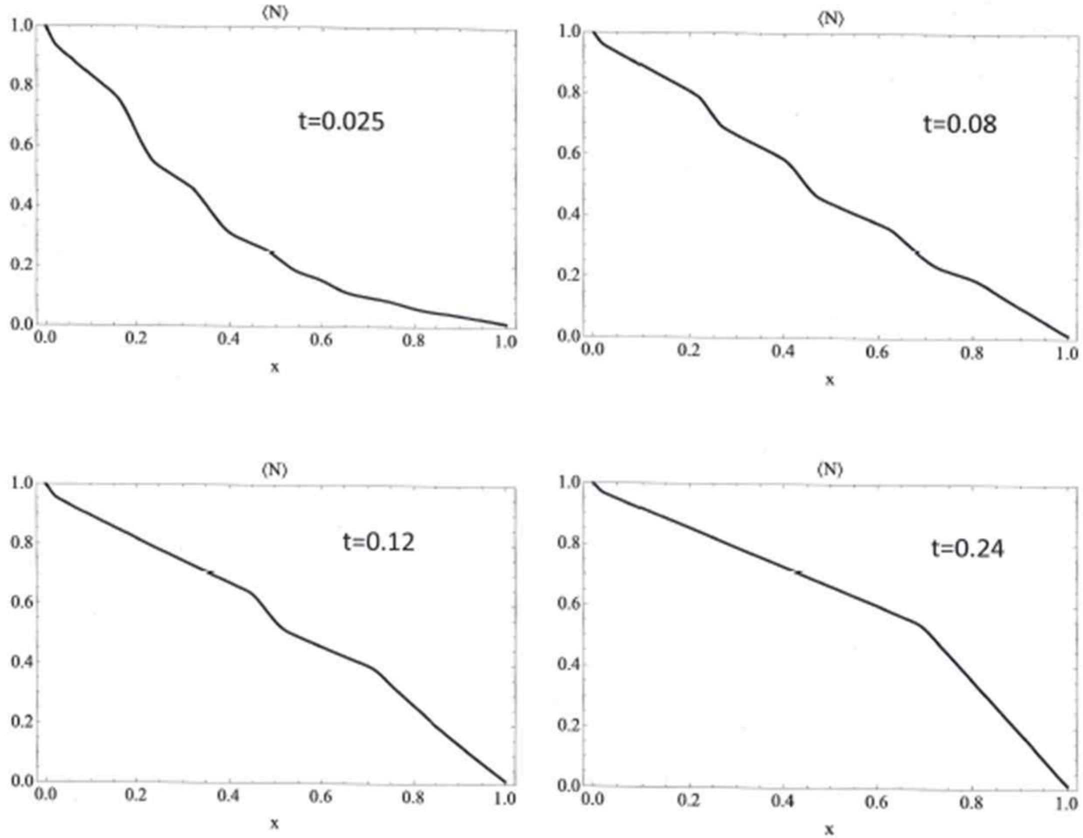


- Single barrier…..

- Quasilinear with l_{mix} feedback

$D_n \neq \chi_y$, $\Pi_{resid} \neq 0$

$\kappa = 2$

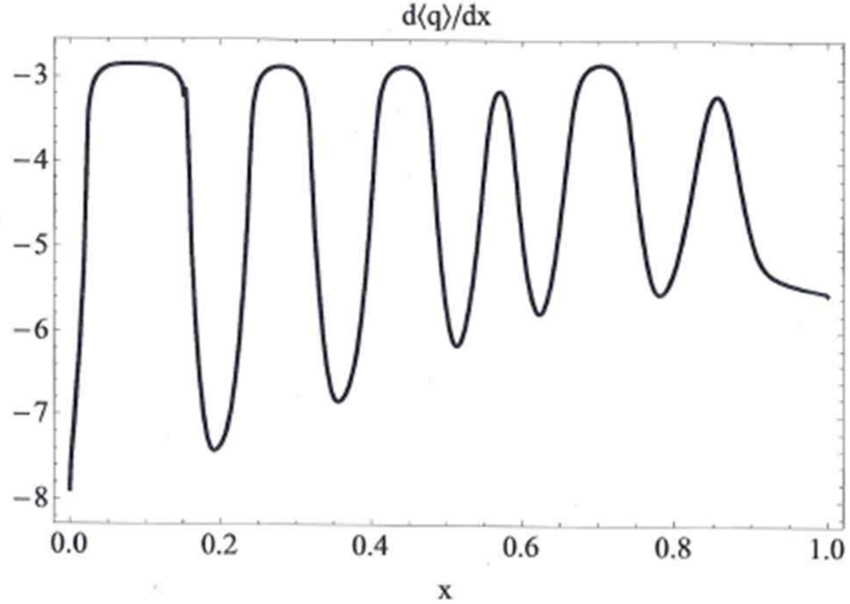


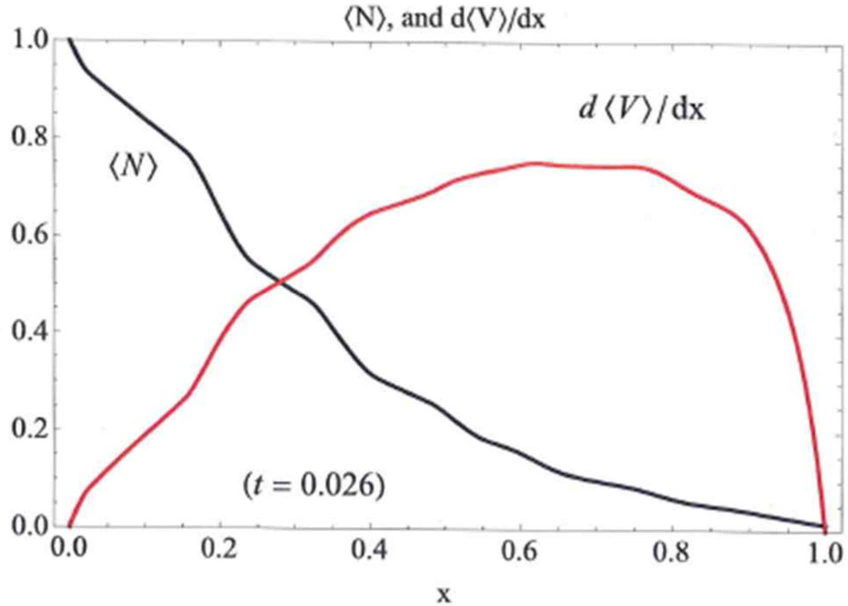
- Density staircase forms and condenses to single edge transport barrier

- Quasilinear with l_{mix} feedback

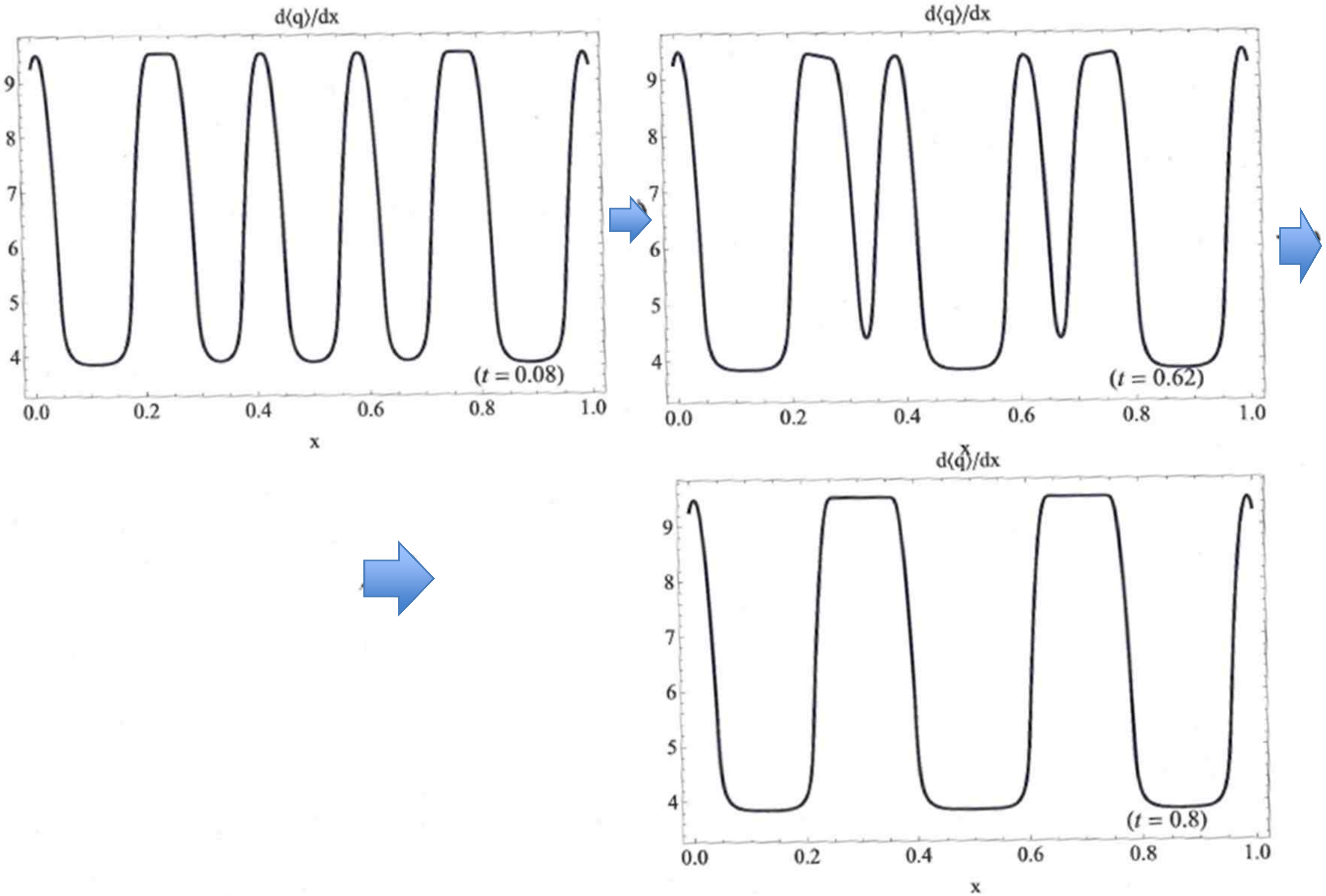
$D_n \neq \chi_y$, $\Pi_{resid} \neq 0$

$\kappa = 2$





- Process of mergers



- What did we learn?
  - Absolutely simplest model recovers staircase
  - Boundary shear layer forms spontaneously
  - * Mergers and propagation down density gradient form macroscopic edge transport barrier from mesoscopic staircase steps!
  - l_{mix} (gradient) feedback seems essential

Discussion

- "Negative diffusion" / clustering instability common to Phillips, QG and DW transport bifurcation and Jam mechanisms:
  - $\delta\Gamma_b/\delta\nabla b < 0$ → Γ_b nonlinearity
  - $\delta\Gamma_q/\ \delta\nabla q < 0$ → $\Gamma_q(\nabla q)$ nonlinearly
- Key elements are:
  - Inhomogeneity in mixing: length scale l_{mix}, and its ∇q dependence, τ_d, etc
  - Feedback loop structure
- * Evidence of step coalescence to form larger scale barriers → pragmatic interest

Areas for further study:

- Structure of mixing representation, form of mixing scales

  → l_{mix}, τ_d

- Non-diffusive flux contributions, form

- Further study of multiple field systems, i.e. H-W:

  $\langle n \rangle, \langle \nabla^2 \phi \rangle, \varepsilon$

- Role of residual transport, spreading

- Step coalescence

- Shear vs PV gradient feedback in QG systems

Final Observation:

Staircases are becoming crowded…

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